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New Developments in the Eight Vertex Model II: Chains of Odd Length

Klaus Fabricius[1] and Barry M. McCoy[2]

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We study the transfer matrix of the 8 vertex model with an odd number of lattice sites N. For systems at the root of unity points $\eta = mK/L$ with m odd the transfer matrix is known to satisfy the famous "TQ" equation where $\mathbf{Q}(v)$ is a specifically known matrix. We demonstrate that the location of the zeroes of this $\mathbf{Q}(v)$ matrix is qualitatively different from the case of even N and in particular they satisfy a previously unknown equation, which is more general than what is often called "Bethe's equation." For the case of even m where no $\mathbf{Q}(v)$ matrix is known we demonstrate that there are many states which are not obtained from the formalism of the SOS model but which do satisfy the TQ equation. The ground state for the particular case of $\eta = 2K/3$ and N odd is investigated in detail.

KEY WORDS: Lattice models; Bethe equations.

1. INTRODUCTION

The eigenvalues of the transfer matrix $\mathbf{T}(v)$ of the eight vertex model with periodic boundary conditions were computed long ago by Baxter.[1,2] This transfer matrix, as given in ref. 2 by

$$\mathbf{T}(u)|_{\mu,\nu} = \mathrm{Tr}\, W(\mu_1, \nu_1) W(\mu_2, \nu_2) \cdots W(\mu_N, \nu_N), \tag{1.1}$$

[1]Physics Department, University of Wuppertal, 42097 Wuppertal, Germany; e-mail: Fabricius@theorie.physik.uni-wuppertal.de
[2]Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840, USA; e-mail: mccoy@insti.physics.sunysb.edu

where $\mu_j, \nu_j = \pm 1$ and $W(\mu, \nu)$ is a 2×2 matrix whose nonvanishing elements are given as

$$\begin{aligned} W(1,1)|_{1,1} &= W(-1,-1)|_{-1,-1} = \rho\Theta(2\eta)\Theta(v-\eta)H(v+\eta) = a(v), \\ W(-1,-1)|_{1,1} &= W(1,1)|_{-1,-1} = \rho\Theta(2\eta)H(v-\eta)\Theta(v+\eta) = b(v), \\ W(-1,1)|_{1,-1} &= W(1,-1)|_{-1,1} = \rho H(2\eta)\Theta(v-\eta)\Theta(v+\eta) = c(v), \\ W(1,-1)|_{1,-1} &= W(-1,1)|_{-1,1} = \rho H(2\eta)H(v-\eta)H(v+\eta) = d(v), \end{aligned} \tag{1.2}$$

has the important property that is satisfies the commutation relation

$$[\mathbf{T}(v), \mathbf{T}(v')] = 0. \tag{1.3}$$

The definition and some useful properties of $H(v)$ and $\Theta(v)$ are summarized in Appendix A.

The solution the 1972 paper[(2)] is restricted to the root of unity condition

$$\eta = mK/L \tag{1.4}$$

and a key property of the computation of the eigenvalues of $\mathbf{T}(v)$ is the definition of an auxiliary matrix $\mathbf{Q}(v)$ with the commutation properties that

$$[\mathbf{T}(v), \mathbf{Q}(v')] = 0, \tag{1.5}$$

$$[\mathbf{Q}(v), \mathbf{Q}(v')] = 0 \tag{1.6}$$

and which satisfies the "TQ" equation

$$\mathbf{T}(v)\mathbf{Q}(v) = [\rho\Theta(0)h(v-\eta)]^N \mathbf{Q}(v+2\eta) + [\rho\Theta(0)h(v+\eta)]^N \mathbf{Q}(v-2\eta), \tag{1.7}$$

where

$$h(v) = \Theta(v)H(v). \tag{1.8}$$

The matrix $\mathbf{Q}_{72}(v)$ defined in Appendix C of ref. 2 was found in ref. 15 (which will be referred to as paper I) to have the quasi-periodicity properties

$$\mathbf{Q}_{72}(v+2K) = \mathbf{S}\mathbf{Q}_{72}(v), \tag{1.9}$$

$$\mathbf{Q}_{72}(v+2iK') = q^{-N}e^{-iN\pi v/K}\mathbf{Q}_{72}(v), \tag{1.10}$$

where

$$\mathbf{S} = \prod_{j=1}^{N} \sigma^{z}. \tag{1.11}$$

The operator S commutes with $\mathbf{T}(v)$ and $\mathbf{Q}_{72}(v)$

$$[\mathbf{T}(v), \mathbf{S}] = [\mathbf{Q}(v), \mathbf{S}] = 0 \tag{1.12}$$

and has eigenvalues ± 1. Therefore we may diagonalize $\mathbf{Q}_{72}(v)$ in the space in which $\mathbf{S}$ is diagonal and in this subspace we see from (1.9) and (1.10) that the eigenvalues $Q_{72}(v)$ of $\mathbf{Q}_{72}(v)$ satisfy

$$Q_{72}(v+2K) = (-1)^{\nu'}Q_{72}(v), \tag{1.13}$$

$$Q_{72}(v+2iK') = q^{-N}e^{-iN\pi v/K}Q_{72}(v). \tag{1.14}$$

We note that under spin inversion $\sigma_j^z \to -\sigma_j^z$ the eigenvalues of the transfer matrix are invariant but that $\mathbf{S} \to (-)^N\mathbf{S}$. When N is odd each eigenvalue of $\mathbf{T}(v)$ is therefore doubly degenerate; one with $\nu' = 0$ and one with $\nu' = 1$. This phenomenon does not happen for N even.

We further found in paper I[15] that for even N if m is even and $N \geqslant L-1$ then $\mathbf{Q}_{72}(v)$ does not exist and that for this case the eigenvalues of $\mathbf{T}(v)$ could only be computed from (1.7) by use of the symmetry property of the eigenvalues of the transfer matrix $T(v;\eta)$ valid for N even

$$T(v+K, K-\eta) = (-1)^{\nu'}T(v,\eta). \tag{1.15}$$

Thus for even N when the root of unity condition (1.4) holds all eigenvalues of $\mathbf{T}(v)$ may be studied by means of the matrix $\mathbf{Q}_{72}(v)$ alone.

An alternative method of computing the eigenvalues of $\mathbf{T}(v)$ which gives in addition the eigenvectors of the transfer matrix was presented by Baxter in 1973.[3–5] In the course of this computation in Section 6 of ref. 3 a new matrix $\mathbf{Q}(v)$ is defined for even N only (for all η not just those satisfying (1.4)) which also has the properties (1.5)–(1.7) found in ref. 2. However, unlike the matrix of the 1972 construction, this new matrix commutes with the spin reflection operator $\mathbf{R}$ which sends the indices μ_j and ν_j into their negatives and instead of the quasi-periodicity conditions (1.9) and (1.10) satisfies the quasiperiodicity conditions (10.5.36a) and (10.5.43.a) of ref. 6

$$\mathbf{Q}_{73}(v+2K) = \mathbf{S}\mathbf{Q}_{73}(v), \tag{1.16}$$
$$\mathbf{Q}_{73}(v+iK') = \mathbf{RS}q^{-N/4}e^{-iN\pi v/2K}\mathbf{Q}_{73}(v). \tag{1.17}$$

In this paper we extend the considerations of paper I to odd N and contrast the construction of the matrix $\mathbf{Q}_{72}(v)$ of Bacter[2] with the construction of eigenvalues and eigenvectors of the transfer matrix $\mathbf{T}(v)$ of refs. 3–5.

For the case of m odd in 1.4 the matrix $\mathbf{Q}_{72}(v)$ defined in ref. 2 exists. In Section 2 we numerically compute the zeroes of the eigenvalues $\mathbf{Q}_{72}(v)$ for odd N and find that they are qualitatively different from the case of N even. We also consider the construction of eigenvectors of refs. 3–5. However this method of computation of eigenvectors requires ((4.2) of ref. 4 and (1.11) of ref. 5) that in order to satisfy the periodic boundary conditions the number of sites N must be of the form

$$N = 2n_B + L_{73}n_L, \tag{1.18}$$

where L_{73} is defined from

$$\eta = 2m_{73}K/L_{73}. \tag{1.19}$$

When m in the root of unity condition (1.4) is odd we see that (1.18) becomes

$$N = 2n_B + 2Ln_L \tag{1.20}$$

which can obviously not hold for odd N and thus the methods of refs. 3–5 are not applicable.

For odd N when m is even or when η is not of the form (1.4) there is no analytic proof of the existence of a matrix $\mathbf{Q}(v)$ with the properties (1.5)–(1.7). Nevertheless since every irrational number can be well approximated by a rational number of the form (1.4) it is of interest to consider (1.7) as an equation for eigenvalues and numerically study the existence of solutions for the eigenvalues $Q(v)$ and to compare these zeroes with the behavior of the zeros computed using the explicit form of $\mathbf{Q}_{72}(v)$. We do this in Section 3.

In Section 4 we study the case where the root of unity condition (1.4) holds with m even. This case is of particular interest because here some of the eigenvalues of the transfer matrix develop extra degeneracies over and beyond the double degeneracy of all eigenvalues which exists for odd

N for all values of η because of spin reflection invariance. Furthermore in this case, even though no matrix $\mathbf{Q}(v)$ is known, the methods of refs. 3–5 allow the computation of (at least some) eigenvectors and eigenvalues of $\mathbf{T}(v)$ provided that (1.18) is satisfied which for even m requires that

$$N = 2n_B + Ln_L \tag{1.21}$$

with n_B and n_L integers. We will see that n_B is the number of pairs of roots

$$v_k^B \text{and } v_k^B + iK' \tag{1.22}$$

of the $Q(v)$ which solves the TQ equation with $T(v)$ and $Q(v)$ considered as scalars and that n_L is the number of L-strings of the form

$$v_k^L + 2jK/L, \quad \text{with } j = 0, \dots, L-1. \tag{1.23}$$

We study these scalar solutions numerically and find that while there are solutions for $Q(v)$ which do have paired roots and L-strings there are also solutions with N roots where there are neither paired roots nor L-strings. The solutions of $Q(v)$ with neither paired roots nor L-strings are not consistent with the restriction (1.21). We therefore conclude that for odd N there are eigenvectors of the transfer matrix $\mathbf{T}(v)$ which cannot be obtained by the methods of refs 3–5. These eigenvectors are significant because they include the ground state of the XYZ spin chain. We study this case analytically for $\eta = 2K/3$ in Section 5.

In Section 6 we study the six vertex limit of the eight vertex model, where the associated spin chain becomes

$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{N}\{\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z\} \tag{1.24}$$

with $\Delta = \cos\pi m/L, \sigma_j^k$ are Pauli spin matrices at site j and periodic boundary conditions are imposed. When m is odd we find that for every $Q(v)$ with $n < N/2$ zeros which satisfies the TQ equation (1.7) there is a second solution with $N - n$ zeros which also satisfies (1.7) with the same eigenvalue of $\mathbf{T}(v)$. This is the extension to rational values of γ of the phenomenon discovered by Pronko and Stroganov[21] for irrational γ and by Bazhanov *et al.*[9,10] in the context of conformal field theory. We contrast our results with the case which has been previously studied[8,11–14] of rational γ with N even and also make contact with the special properties which the value $\Delta = -1/2$ has for odd N.[17,22,23] We conclude with a discussion of the significance of our results in Section 7.

2. ODD N FOR $\eta = MK/L$ WITH M ODD AND L EVEN OR ODD

When N is odd with m odd and L even or odd then the restriction (1.20) cannot be satisfied but the matrix $\mathbf{Q}_{72}(v)$ exists. Thus in this case we may proceed as we did in the case of even N in paper I[(15)] and numerically determine the zeroes of the eigenvalues of $\mathbf{Q}_{72}(v)$ directly from the definition of ref. 2.

The most general function which satisfies the quasi-periodicity properties (1.13) and (1.14) can be written in the factorized form

$$Q_{72}(v) = \mathcal{K}(q, v_k)\exp(-i\nu\pi v/2K)\prod_{j=1}^{N} H(v - v_j), \tag{2.1}$$

where the v_j are unique once we adopt a convention for the location of the fundamental region of the quasiperiodic function $H(v)$. Substituting (2.1) in (1.13) and using (A4) we find

$$e^{i\pi(\nu' + \nu + N)} = 1 \quad \text{so } \nu' + \nu + N = \text{even integer} \tag{2.2}$$

and substituting (2.1) in (1.14) and using (A5) we find

$$\begin{aligned} &e^{\pi i(-i\nu K'/K + N + \sum_{j=1}^{N} v_j/K)} = 1 \\ &\text{so } N + \left(-\nu i K' + \sum_{j=1}^{N} v_j\right)/K = \text{even integer.} \end{aligned} \tag{2.3}$$

Taking real and imaginary parts of (2.3) we obtain the sum rule

$$N + \sum_{j=1}^{N} \mathrm{Re}(v_j)/K = \text{even integer} \tag{2.4}$$

and find that ν satisfies

$$\nu = \sum_{j=1}^{N} \mathrm{Im}(v_j)/K' = \text{even integer} - \nu' - N, \tag{2.5}$$

where the value of the even integer and thus the numerical value of ν depends on the conventions used to specify the fundamental region for v_j.

We have numerically studied the zeros of all the eigenvalues $Q_{72}(v)$. For the case of even N we found in paper I[15] that the zeros either occur in pairs (1.22) which we call Bethe roots or L-strings (1.23). Therefore for even N all the eigenvalues $Q_{72}(v)$ may be written as

$$Q_{72}(v) = \mathcal{K}(q, v_k)\exp(-i\nu\pi v/2K)\prod_{j=1}^{n_B} H(v - v_j^B)H(v - v_j^B - iK')$$
$$\times\prod_{j=1}^{n_L} H(v - v_j^L)H(v - v_j^L - 2K/L)\cdots H(v - v_j^L - 2(L-1)K/L), \tag{2.6}$$

where

$$2n_B + Ln_L = N. \tag{2.7}$$

Using (A8) we may rewrite this as

$$Q_{72}(v) = \tilde{\mathcal{K}}(q, v_k)\exp(-i(\nu - n_B)\pi v/2K)(-i)^{n_B} q^{-n_B/4}\prod_{j=1}^{n_B} h(v - v_j^B)$$
$$\times\prod_{j=1}^{n_L} H(v - v_j^L)H(v - v_j^L - 2K/L)\cdots H(v - v_j^L - 2(L-1)K/L), \tag{2.8}$$

where

$$\tilde{\mathcal{K}}(q, v_k) = \mathcal{K}(q, v_k)e^{-\pi i\sum_{j=1}^{n_B} v_j^B/(2K)}. \tag{2.9}$$

The roots v_j^B are called Bethe roots and by substitution of (2.8) into the TQ equation (1.7) with v set equal to v_j^B we see that the v_j^B satisfy the "Bethe's" equation

$$\left(\frac{h(v_l^B - \eta)}{h(v_l^B + \eta)}\right)^N = e^{2\pi i(\nu - n_B)m/L}\prod_{\substack{j=1\\ j\neq l}}^{n_B}\frac{h(v_l^B - v_j^B - 2\eta)}{h(v_l^B - v_j^B + 2\eta)}. \tag{2.10}$$

The roots v_j^L are determined by the new functional equation conjectured in I[15]

$$\mathbf{A}'e^{-N\pi iv/2K}\mathbf{Q}_{72}(v - iK') = \sum_{l=0}^{L-1}\frac{h^N(v - (2l+1)\eta)\mathbf{Q}_{72}(v)}{\mathbf{Q}_{72}(v - 2l\eta)\mathbf{Q}_{72}(v - 2(l+1)\eta)}, \tag{2.11}$$

where $\mathbf{A}'$ is a matrix (called $\mathbf{A}^{-1}$ in ref. 15) which commutes with $\mathbf{Q}_{72}(v)$, is independent of v and depends on the normalization in the construction of $\mathbf{Q}_{72}(v)$.

The present case when N is odd is quite different from the case N even previously studied. Now our numerical study finds that there are neither the paired solutions (1.22) nor the L string solutions (1.23). Instead of the pairing condition of roots (1.22) we now find that there is a pairing of solutions in the sense that for every set of roots v_j there is a second set of roots v_j+iK', which also gives an eigenvalue of $\mathbf{Q}_{72}(v)$ and these two paired eigenvalues of $\mathbf{Q}_{72}(v)$ satisfy the TQ equation with the same eigenvalue of $\mathbf{T}(v)$ Noting that

$$\sum_{j=1}^{N}\mathrm{Im}(v_j+iK')/K'=N+\sum_{j=1}^{N}\mathrm{Im}(v_j)/K' \tag{2.12}$$

we conclude from (2.5) that when N is odd the parity of ν' for the shifted solution is opposite to the parity of ν' for the original solution. Thus since shifting all roots by iK' is equivalent to sending $v\to iK'$ we conclude that

$$Q_{72}(v+iK',\nu'=0)=Q_{72}(v,\nu'=1). \tag{2.13}$$

We obtain an equation for the N roots v_j by using the factorized form (2.1) for $Q_{72}(v)$ and set $v=v_j$ in the TQ equation (1.7) to obtain for odd N with m odd

$$\left(\frac{h(v_l-mK/L)}{h(v_l+mK/L)}\right)^N=e^{2\pi i\nu m/L}\prod_{\substack{j=1\\ j\neq l}}^{N}\frac{H(v_l-v_j-2mK/L)}{H(v_l-v_j+2mK/L)}, \tag{2.14}$$

which differs from the Bethe's equation (2.10) for even N in that the function $H(v)$ appears on the right hand side instead of $h(v)$ and the number of terms in the product is N instead of n_B. This equation seems to be new in the literature.

3. ODD N AND GENERIC η

For odd N and η not of the form (1.4) with m odd there is no analytic proof that there is a matrix $\mathbf{Q}(v)$ with the properties (1.5) and (1.6) which satisfies the TQ equation (1.7). However for finite N if L is sufficiently large it should be impossible to tell the difference between values of η which are generic and those which satisfy (1.4) with m odd. Consequently instead of computing the zeroes of $Q(v)$ using the specific form of $\mathbf{Q}_{72}(v)$ given in ref. 2 we may consider (1.7) as an equation for eigenvalues, and see numerically if for each eigenvalue $T(v)$ it is possible to find a $Q(v)$ which satisfies (1.7). We have made such a study for $N=7$ and $N=9$ at $q=0.2$ and find that for each eigenvalue of $\mathbf{T}(v)$ there do indeed exist two distinct functions $Q(v)$ and $Q(v+iK')$ which satisfy the TQ equation (1.7). These functions $Q(v)$ share with the eigenvalues of $\mathbf{Q}_{72}(v)$ the property of having N roots and these roots satisfy the sum rules (2.4) and (2.5) which follow from the quasiperiodicity conditions

$$Q(v+2K) = (-1)^{\nu'} Q(v), \tag{3.1}$$
$$Q(v+2iK') = q^{-N} e^{-iN\pi v/K} Q(v), \tag{3.2}$$

which generalize the quasiperiodicity conditions (1.13) and (1.14) of the eigenvalues of $\mathbf{Q}_{72}(v)$. The N roots v_k satisfy the generalization of (2.14) to generic values of eta

$$\left(\frac{h(v_l-\eta)}{h(v_l+\eta)}\right)^N = e^{2\pi i\nu\eta/K} \prod_{\substack{j=1 \\ j\neq l}}^{N} \frac{H(v_l-v_j-2\eta)}{H(v_l-v_j+2\eta)}. \tag{3.3}$$

We refer to (3.3) as the generic equation for roots.

We give in Table I an example of the relation of the zeroes of two eigenvalues of $Q(v)$ computed for $N=9$ at $\eta=K/3$ from $\mathbf{Q}_{72}(v)$ and computed numerically at $\eta=0.35K$ from the TQ equation.

4. ODD N FOR $\eta=M/L$ WITH M EVEN AND L ODD

When η satisfies the root of unity condition (1.4) with m even and when N satisfies $N=2n_B+Ln_L$ (some of) the eigenvectors and eigenvalues of the transfer matrix $\mathbf{T}(v)$ may be computed by the methods of refs 3–5. As with the case of generic η there is no analytic proof in this case of the existence for odd N of a matrix $\mathbf{Q}_o(v)$, which satisfies (1.5)–(1.7) and to gain insight we numerically solve the TQ equation (1.7) for scalar functions $Q(v)$ for (1) $\eta=2K/3$ and a nearby value $\eta=0.65K$ for $N=9$ and $q=0.2$ and for (2) $\eta=2K/5$ and a nearby value of $\eta=0.398$ for $N=7$ and $q=0.2$ For $\eta=0.65K$ and $0.398K$ the numerical solutions have the feature of generic values of η that they have N roots which satisfy the sum rules (2.4) and (2.5). However, at precisely $\eta=2K/3$ and $\eta=2K/5$ there is qualitative change in the numerical solutions. For $\eta=2K/3$ there are only two

Table I. A Comparison of the Roots of Eigenvalues of $Q_{72}(v)$ at $\eta=K/3$ with Numerical Solutions to the TQ Equation at $\eta=0.35K$ for the Nome $q=0.20$

State	$Q(v)$ $\eta=0.35\,\mathrm{K}$	$Q_{72}(v)$ $\eta=K/3$
1	$i0.06866174K'$	$i0.06598354K'$
	$i0.57652164K'$	$i0.57706877K'$
	$i0.83871494K'$	$i0.84250034K'$
	$i0.97126416K'$	$i0.97027666K'$
	$i1.08588486K'$	$i1.08005865K'$
	$i1.22542023K'+K$	$i1.21577690K'+K$
	$i1.48883198K'$	$i1.49156270K'$
	$i1.79747242K'$	$i1.80675767K'$
	$i1.94722804K'$	$i1.95008384K'$
2	$i1.06866174K'$	$i1.06598354K'$
	$i1.57652164K'$	$i1.57706877K'$
	$i1.83871494K'$	$i1.84250034K'$
	$i1.97126416K'$	$i1.97027666K'$
	$i0.08588486K'$	$i0.08005865K'$
	$i0.22542023K'+K$	$i0.21577690K'+K$
	$i0.48883198K'$	$i0.49156270K'$
	$i0.79747242K'$	$i0.80675767K'$
	$i0.94722804K'$	$i0.95008384K'$
3	$i0.09682452K'+K$	$i0.09803186K'+K$
	$i0.31149066K'+K$	$i0.31332654K'+K$
	$i0.56470429K'+K$	$i0.56389029K'+K$
	$i0.80075993K'+K$	$i0.79896112K'+K$
	$iK'+K$	$iK'+K$
	$i1.19924006K'+K$	$i1.20103887K'+K$
	$i1.43529570K'+K$	$i1.43610970K'+K$
	$i1.68850933K'+K$	$i1.68667345K'+K$
	$i1.90304175K'+K$	$i1.90196813K'+K$
4	$i1.09682452K'+K$	$i1.09803186K'+K$
	$i1.31149066K'+K$	$i1.31332654K'+K$
	$i1.56470429K'+K$	$i1.56389029K'+K$
	$i1.80075993K'+K$	$i1.79896112K'+K$
	$iK'+K$	$iK'+K$
	$i0.19924006K'+K$	$i0.20103887K'+K$
	$i0.43529570K'+K$	$i0.43610970K'+K$
	$i0.68850933K'+K$	$i0.68667345K'+K$
	$i0.90304175K'+K$	$i0.90196813K'+K$

The roots of state 2 and 4 are shifted from those of states 1 and 3 by $iK'(\text{mod } 2iK')$. States 3 and 4 are the two degenerate ground states. The roots are accurate to the number of significant figures given.

solutions which have N roots and while for $\eta = 2K/5$ for $N = 7$ the number of solutions with 7 roots decreases from 128 to 58. All other solutions have less than N roots and they do not satisfy the sum rules (2.4) and (2.5). We illustrate this in Tables II and IV for $\eta = 2K/3$ and in Tables V and VI for $\eta = 2K/5$.

What is clearly visible in the examples of Tables II–V (and is seen in all the rest of the data) is that as $\eta \to 2mK/L$ two phenomena occur:

(1) There are $n_L L$ roots which move to form n_L L strings where L roots satisfy

$$\alpha_k, \ \ \alpha_k + 2\eta, \ldots, \alpha_k + 2(L-1)\eta \tag{4.1}$$

and make a contribution to $Q_o(v)$ of

$$\prod_{k=1}^{n_L} \prod_{j=0}^{L-1} H(v - \alpha_k - 2j\eta), \tag{4.2}$$

which cancels out of (2.14).

(2) The remaining $n_B = (N - n_L L)/2$ roots arrange themselves in pairs v_k and $v_k + iK.'$

Taking into account both of these phenomena we see that the generic equation for the roots (2.14) reduces to the "Bethe's" equation (2.10) derived in refs. 3–5. However, the roots of those $Q_o(v)$ which do not have paired roots and L strings as $\eta \to 2mK/L$ satisfy the generic equation (2.14) instead of the Bethe's equation (2.10). We thus conclude from these numerical studies that for odd N with m even there exist eigenvalues of the transfer matrix, which cannot be obtained by the methods of refs. 3–5. For these states the eigenvalues $Q_o(v)$ of the TQ equation (1.7) have been found numerically to satisfy

$$\sum_{l=0}^{L-1} \frac{h^N(v - (2l+1)\eta)}{Q_o(v - 2l\eta)Q_o(v - 2(l+1)\eta)} = 0. \tag{4.3}$$

If there is one L string in the eigenvalue of $\mathbf{Q}_o(v)$ the string center of the limiting values drops out of the TQ equation but may be computed from the $N - L$ roots which do satisfy the generic equation (3.3) by means of the sum rule (2.5). When there are 3 or more L strings equations for the string centers may be obtained by again carefully taking the limit of the generic equation (3.3) for the roots of $Q_o(v)$ by generalizing the corresponding six vertex computation of ref. 12 to find

Table II. A Comparison of the Roots of Numerical Solutions of the TQ Equation for $\eta = 0.65K$ and $\eta = 2K/3$ for $N = 9$ and $q = 0.2$

State	$\eta = 0.65\,\mathrm{K}$	$\eta = 2K/3$
1	$i0.1548264K'$	$i0.14488580K'$
	$i0.40419528K'$	$i0.39698346K'$
	$i0.92299060K'$	$i0.92807395K'$
	$i1.13102864K'$	$i1.14488580K'$
	$i1.37507482K'$	$i1.39698346K'$
	$i1.94364493K'$	$i1.92807395K'$
	$i1.68925695K' + 0.27932034K$	
	$i1.68925695K' + 1.72067964K$	
	$i1.68972529K' + K$	
2	$i1.1548264K'$	$i1.14488580K'$
	$i1.40419528K'$	$i1.39698346K'$
	$i1.92299060K'$	$i1.92807395K'$
	$i0.13102864K'$	$i0.14488580K'$
	$i0.37507482K'$	$i0.39698346K'$
	$i0.94364493K'$	$i0.92807395K'$
	$i0.68925695K' + 0.27932034K$	
	$i0.68925695K' + 1.72067964K$	
	$i0.68972529K' + K$	
3	$i0.06421637K' + K$	$i0.05992394K' + K$
	$i0.23183303K' + K$	$i0.21760180K' + K$
	$i0.61330441K' + K$	$i0.62586149K' + K$
	$i0.86288322K' + K$	$i0.89718639K' + K$
	$iK' + K$	$iK' + K$
	$i1.13711677K' + K$	$i1.12813602K' + K$
	$i1.38669558K' + K$	$i1.37413850K' + K$
	$i1.76816695K' + K$	$i1.78239820K' + K$
	$i1.93568362K' + K$	$i1.94007605K' + K$
4	$i1.06421637K' + K$	$i1.05992394K' + K$
	$i1.23183303K' + K$	$i1.21760180K' + K$
	$i1.61330441K' + K$	$i1.62586149K' + K$
	$i1.86288322K' + K$	$i1.89718639K' + K$
	K	K
	$i0.13711677K' + K$	$i0.12813602K' + K$
	$i0.38669558K' + K$	$i0.37413850K' + K$
	$i0.76816695K' + K$	$i0.78239820K' + K$
	$i0.93568362K' + K$	$i0.94007605K' + K$

In states 1 and 2 a three string develops as $\eta \rightarrow 2K/3$ which drops out of the TQ equation at $\eta = 2K/3$. At $\eta = 2K/3$ the six roots obtained from the TQ equation occur in pairs v_k and $v_k + iK'$. States 3 and 4 are the two degenerate ground states which have no L strings. The roots are accurate to the number of significant figures given.

Table III. Examples of the Development of the Octet of States with 3 L Strings for $N=9$ as $\eta \to 2K/3$ with Momentum $P=-2\pi/3$

State	$\eta = 0.666\,\mathrm{K}$	H_{XYZ}
1	$i0.10731398K'$	-4.7526939
	$i0.10854909K' + .66600000K$	
	$i0.10854909K' + 1.3339999K$	
	$i0.56699150K' + K$	
	$i0.56696222K' + 0.33190487K$	
	$i0.56696222K' + 1.66809512K$	
	$i0.99234471K'$	
	$i0.99120488K' + 0.66600000K$	
	$i0.99120488K' + 1.33400000K$	
2	$i0.19439140K' + K$	-4.7484930
	$i0.19440408K' + 0.33199252K$	
	$i0.19440408K' + 1.66800747K$	
	$i0.65737246K'$	
	$i0.65549419K' + 0.66600875K$	
	$i0.65549419K' + 1.33399124K$	
	$i1.81540681K'$	
	$i1.81651638K' + 0.66600012K$	
	$i1.81651638K' + 1.33399987K$	
3	$i0.19872733K' + K$	-4.7487136
	$i0.19874105K' + 0.33199370K$	
	$i0.19874105K' + 1.66800629K$	
	$i0.90240897K'$	
	$i0.90240470K' + 0.33199551K$	
	$i0.90240470K' + 1.66800448K$	
	$i1.56554439K' + K$	
	$i1.56551388K' + 0.33190519K$	
	$i1.56551388K' + 1.66809480K$	
4	$i0.47511066K'$	-4.7457518
	$i0.47701987K' + 0.66597664K$	
	$i0.47701987K' + 1.33402335K$	
	$i0.90529517K' + K$	
	$i0.90291287K' + 0.33199520K$	
	$i0.90291287K' + 1.66800480K$	
	$i1.28586937K'$	
	$i1.28455121K' + 0.66599732K$	
	$i1.28455121K' + 1.33400267K$	

The numerical solutions of the TQ equation are given for $\eta = 0.666K$. The values of the corresponding XYZ spin chain are shown to illustrate the approach to degeneracy. Only 4 of the members of the octet are given. The remaining 4 are obtained by sending $v_k \to v_k + iK'$ (mod $2iK'$). The roots for corresponding state with $P=2\pi/3$ are the complex conjugates (mod 2iK') of the roots for $P=-2K/3$. The roots are accurate to the number of significant figures given.

Table IV. Examples of the Development of the Octet of States with 3 L Strings for $N=9$ as $\eta \to 2K/3$ with Momentum $P=0$

State	$\eta = 0.666\,\mathrm{K}$	H_{XYZ}
1	K $0.33199540K$ $1.66800459K$ $i0.69090441K' + K$ $i0.69087039K' + 0, 33197194K$ $i0.69087039K' + 1.66802805K$ $i1.30955843K' + K$ $i1.30912960K' + 0.33197194K$ $i1.30912960K' + 1.66802805K$	−4.7484601
2	K $0.33199539K$ $1.66800460K$ $i0.39752759K'$ $i0.39589406K' + 0.66601414K$ $i0.39589406K' + 1.33398585K$ $i1.60247240K'$ $i1.60410593K' + 0.66601414K$ $i1.60410593K' + 1.33398585K$	−4.7440296
3	$i0.30522748K' + K$ $i0.30525934K' + 0.33197051K$ $i0.30525934K' + 1.66802948K$ $i0.78313587K'$ $i0.78155612K' + 0.66599948K$ $i0.78155612K' + 1.33400051K$ $i1.91196073K'$ $i1.91302248K' + 0.66599999K$ $i1.91302248K' + 1.33400000K$	−4.7511419
4	$i1.69477251K' + K$ $i1.69474065K' + 0.33197051K$ $i1.69474065K' + 1.33802948K$ $i1.21686441K'$ $i1.21844389K' + 0.66599948K$ $i1.21844389K' + 1.33400051K$ $i0.08803926K'$ $i0.086977515K' + 0.66599999K$ $i0.086977515K' + 1.33400000K$	−4.7511419

The numerical solutions of the TQ equation are given for $\eta = 0.666K$. The values of the corresponding XYZ spin chain are shown to illustrate the approach to degeneracy. Only four of the members of the octet are given. The remaining four are obtained by sending $v_k \to v_k + iK'$ (mod $2iK'$). The roots in states 1 and 2 are invariant under complex conjugations (mod $2iK'$) and states 3 and 4 transform into each other under complex conjugation. The roots are accurate to the number of significant figures given.

Table V. A Comparison of the Roots of Numerical Solutions of the TQ Equation for $\eta = 0.398\,\mathrm{K}$ and $\eta = 2K/5$ for $N = 7$ and $q = 0.2$ for States which do not Develop L Strings or Pairs v_k and $v_k + iK'$

State	$\eta = 0.398\,\mathrm{K}$	$\eta = 2K/5$
1	$i0.04397370K'$	$i0.04471961K'$
	$i0.20176882K' + K$	$i0.20257510K' + K$
	$i0.46160907K'$	$i0.46092396K'$
	$i0.81633633K'$	$i0.81538032K'$
	$i1.00694470K'$	$i1.00687865K'$
	$i1.59024152K'$	$i1.59035264K'$
	$i1.87912581K'$	$i1.87917007K'$
2	$i1.04397370K'$	$i1.04471961K'$
	$i1.20176882K' + K$	$i1.20257510K' + K$
	$i1.46160907K'$	$i1.46092396K'$
	$i1.81633633K'$	$i1.81538032K'$
	$i0.00694470K'$	$i0.00687865K'$
	$i0.59024152K'$	$i0.59035264K'$
	$i0.87912581K'$	$i0.87917007K'$
3	$i0.2553638K' + K$	$i0.25505369K' + K$
	$i0.58661122K' + K$	$i0.58678614K' + K$
	$i0.87889895K' + K$	$i0.87909858K' + K$
	$iK' + K$	$iK' + K$
	$i1.12110104K' + K$	$i1.12090141K' + K$
	$i1.41338877K' + K$	$i1.41321385K' + K$
	$i1.74463612K' + K$	$i1.74494630K' + K$
4	$i1.2553638K' + K$	$i1.25505369K' + K$
	$i1.58661122K' + K$	$i1.58678614K' + K$
	$i1.87889895K' + K$	$i1.87909858K' + K$
	K	K
	$i0.12110104K' + K$	$i0.12090141K' + K$
	$i0.41338877K' + K$	$i0.41321385K' + K$
	$i0.74463612K' + K$	$i0.74494630K' + K$

States 3 and 4 are the two degenerate ground states. The roots are accurate to the number of significant figures given.

$$\sum_{k=1}^{L} \frac{c_0^{-(k+1)} \Phi_{k+1}(\alpha_j)}{P_k(\alpha_j) P_{k+1}(\alpha_j)}$$
$$\times \left[N \frac{h'(\alpha_j + (2k+1)\eta_0)}{h(\alpha_j + (2k+1)\eta_0)} - \sum_{i=1}^{n_0} \left(\frac{H'(\alpha_j - v_i^{(0)} + (k+1)2\eta_0)}{H(\alpha_j - v_i^{(0)} + (k+1)2\eta_0)} + \frac{H'(\alpha_j - v_i^{(0)} + k2\eta_0)}{H(\alpha_j - v_i^{(0)} + k2\eta_0)} \right) \right]$$
$$-2K(\alpha_j) \sum_{l \neq j} \sum_{m=1}^{L} \frac{H'(\alpha_j - \alpha_l + m2\eta_0)}{H(\alpha_j - \alpha_l + m2\eta_0)} + K(\alpha_j) \frac{i\pi\nu}{K} = 0, \tag{4.4}$$

Table VI. A Comparison of the Roots of Numerical Solutions of the TQ Equation for $\eta = 0.398K$ and $\eta = 2K/5$ for $N = 7$ and $q = 0.2$ for States which do Develop Five Strings and Paired Roots

State	$\eta = 0.398\,\mathrm{K}$	$\eta = 2K/5$
1	$i0.76120023K'$	$i0.76089181K'$
	$i1.75808435K'$	$i1.76089181K'$
	$i1.48754767K' + K$	
	$i1.49676120K' + 1.75698935K$	
	$i1.49676120K' + 0.24301064K$	
	$i1.49982265K' + 1.44611774K$	
	$i1.49982265K' + 0.55388225K$	
2	$i1.76120023K'$	$i0.76089181K'$
	$i0.75808435K'$	$i1.76089181K'$
	$i0.48754767K' + K$	
	$i0.49676120K' + 1.75698935K$	
	$i0.49676120K' + 0.24301064K$	
	$i0.49982265K' + 1.44611774K$	
	$i0.49982265K' + 0.55388225K$	

The roots are accurate to the number of significant figures given.

where $v_i^{(0)}$ are the ordinary Bethe roots which occur in pairs, $j = 1, 2, \ldots, n_L$,

$$c_0 = \exp(2\pi i \nu\eta/K) \tag{4.5}$$

$$P_k(\alpha_j) = \prod_i^{n_0} H(\alpha_j - v_i^{(0)} + k2\eta_0), \tag{4.6}$$

$$\Phi_k(\alpha_j) = h^N(\alpha_j + (2k-1)\eta_0) \tag{4.7}$$

and

$$K(\alpha_j) = \sum_{k=1}^{L} \frac{c_0^{-(k+1)}\Phi_{k+1}(\alpha_j)}{P_k(\alpha_j)P_{k+1}(\alpha_j)}, \tag{4.8}$$

5. THE GROUND STATE FOR ODD N and $\eta = 2K/3$

To gain insight into these solutions which have the pairing property and are not obtained from refs. 3–5 we consider the simplest case $\eta = 2K/3$ and recall that in this case Baxter[7] has, by means of the inversion

Table VII. Roots of $Q(v)$ for the Six Vertex Model with $\Delta = +1/2$ with $N = 9$ Illustrating the Pairing of Solutions with n and $N - n$ Roots

$n=0$	$n=9$	E_{XXZ}	P
none	$\pm i0.74839\ldots$ 0 $\pm i1.52761\cdots+\pi/2$ $\pm i0.3.154723\cdots\pm 1.047189\cdots$	-2.25	0
$n=1$	$n=8$		
0	$\pm i1.261399\cdots$ $\pm i0.281540\cdots$ $\pm i0.636873\cdots\pm 1.048336\cdots$	$-3.25\cdots$	0
$-i0.742104\cdots$	$i1.220370\cdots$ $i0.228906\cdots$ $-i0.361558\cdots$ $-i1.401228\cdots$ $\pm i0.591172\cdots\pm 1.048234\cdots$	$-2.78208\cdots$	$2\pi/9$
$-i0.844260\cdots+\pi/2$	$i1.163039\cdots$ $i0.183634\cdots$ $-i0.616811\cdots$ $-i0.737347\cdots+\pi/2$ $\pm i0.559024\cdots\pm 1.0480858\cdots$	$-1.59729\cdots$	$4\pi/9$
$-i0.346573\cdots+\pi/2$	$i1.176586\cdots$ $i0.126324\cdots$ $-i1.060573$ $-i0.359837\cdots+\pi/2$ $\pm i0.528842\cdots\pm 1.047924\cdots$	-0.25	$6\pi/9$
$-i0.102156\cdots+\pi/2$	$i1.155571\cdots$ $i0.0486462\cdots$ $-i1.128327\cdots$ $-i0.983179\cdots+\pi/2$ $\pm i0.4920739\cdots\pm 1.045727\cdots$	$0.629385\cdots$	$8\pi/9$

Each of the paired solutions corresponds to the same eigenvalue of the six vertex transfer matrix and XXZ spin chain.The energy eigenvalue E_{XXZ} of H_{XXZ} and the corresponding momentum P are given and the $\pm$ are to be taken independently. The roots for the four negative values of P are the complex conjugates of roots for positive P.

relation found that the transfer matrix has an exact (doubly degenerate) eigenvalue

$$T(v) = (a(v) + b(v))^N = [\rho\Theta(0)h(v)]^N. \tag{5.1}$$

This is the eight vertex generalization of the six vertex problem at $\Delta = -1/2$ considered in refs. 17,22,23.

The TQ equation (1.7) for the eigenvalue of $\mathbf{T}(v)$ given by (5.1) is

$$h(v)^N Q_{N,\nu'}(v) = h(v-2K/3)^N Q_{N,\nu'}(v+4K/3) \\ +h(v+2K/3)^N Q_{N,\nu'}(v-4K/3). \tag{5.2}$$

Using (A4) and the fact that N is odd (5.2) is rewritten as

$$h(v)^N Q_{N,\nu'}(v) = -h(v+4K/3)^N Q_{N,\nu'}(v+4K/3) \\ -h(v-4K/3)^N Q_{N,\nu'}(v-4K/3). \tag{5.3}$$

It is easy to see that if $Q_{N,\nu'}(v)$ satisfies (5.3) that $Q_{N,\nu'}(-v)$ will satisfy the equation also. Thus we can require that our two independent solutions satisfy

$$Q^e_{N,\nu'}(-v) = Q^e_{N,\nu'}(v), \tag{5.4}$$
$$Q^o_{N,\nu'}(-v) = -Q^o_{N,\nu'}(v). \tag{5.5}$$

The odd solution, however, cannot occur. This can be seen if we note that both $Q^e_{N,\nu'}(v)$ and $Q^o_{N,\nu'}(v)$ can be written as

$$Q^{v_0}_{N,\nu'}(v) = e^{-i\nu\pi v/2K} H(v-v_0) \prod_{j=1}^{(N-1)/2} H(v-v_j)H(v+v_j) \tag{5.6}$$

with v_j purely imaginary and an appropriate choice of v_0. From the quasi-periodicity of $Q^{v_0}_{N,\nu'}$ it follows that the sum rule (2.4) holds and thus we find that

$$\mathrm{Re}(v_0) = K \tag{5.7}$$

and similarly the quasiperiodicity requires (2.5) to hold and thus we find that either

$$\mathrm{Im}(v_0) = K', \quad \nu' = 0, \quad \nu = 1 \quad \text{or}, \tag{5.8}$$
$$\mathrm{Im}(v_0) = 0, \quad \nu' = 1, \quad \nu = 0. \tag{5.9}$$

For either (5.8) or (5.9) it follows from the properties (A4) and (A5) that for $v_0 = 0, iK'$ that $Q^{v_0}_{N,\nu'}(v)$ is even and thus $Q^o_{N,\nu'}(v)$ does not occur.

We now define

$$f_{N,\nu'}(v) = h(v)^N Q_{N,\nu'}(v) \tag{5.10}$$

and thus (5.3) becomes

$$f_{N,\nu'}(v) + f_{N,\nu'}(v - 4K/3) + f_{N,\nu'}(v + 4K/3) = 0. \tag{5.11}$$

From the quasi-periodicity properties (3.1) and (3.2) and the evenness of $Q_{N,\nu'}(v)$ we see that $f_{N,\nu'}(v)$ must also satisfy

$$f_{N,\nu'}(v + 2K) = (-)^{1+\nu'} f_{N,\nu'}(v), \tag{5.12}$$
$$f_{N,\nu'}(v + 2iK') = q^{-3N} e^{-3N\pi i v/K} f_{N,\nu'}(v), \tag{5.13}$$
$$f_{N,\nu'}(-v) = -f_{N,\nu'}(v). \tag{5.14}$$

We satisfy (5.12) and (5.14) by writing

$$f_{N,\nu'}(v) = \sum_{j=-\infty}^{\infty} \tilde{f}_{j,\nu'} e^{i\pi(j+\frac{1+\nu'}{2})v/K} \tag{5.15}$$

with

$$\tilde{f}_{-j,\nu'} = -\tilde{f}_{j-1-\nu',\nu'} \tag{5.16}$$

and find from (5.13) that

$$\tilde{f}_{j+3N,\nu'} = q^{3N+2j+1+\nu'} \tilde{f}_{j,\nu'}. \tag{5.17}$$

Thus defining

$$a_{j,\nu'} = q^{-(j+\frac{1+\nu'}{2})^2/3N} \tilde{f}_{j,\nu'} \tag{5.18}$$

we have

$$a_{j+3N,\nu'} = a_{j,\nu'}, \qquad a_{-j',\nu'} = -a_{j-1-\nu',\nu'} \tag{5.19}$$

and

$$f_{N,\nu'}(v)=\sum_{j=-\infty}^{\infty} a_{j,\nu'}q^{(j+\frac{1+\nu'}{2})^2/3N}e^{i\pi(j+\frac{1+\nu'}{2})v/K}. \tag{5.20}$$

Setting $j=r3N+l$ and using (5.19) in (5.20) we find

$$\begin{aligned} f_{N,\nu'}(v) = &\sum_{l=0}^{3(N-1)/2} a_{l,\nu'} \sum_{r=-\infty}^{\infty} q^{3N(r+\frac{1+\nu'+2l}{6N})^2} \\ &\times[e^{i\pi(r+\frac{1+\nu'+2l}{6N})3Nv/K}-e^{-i\pi(r+\frac{1+\nu'+2l}{6N})3Nv/K}], \end{aligned} \tag{5.21}$$

which is expressed in terms of the standard theta function with characteristics $\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](v,\tau)$, whose definition and some useful properties are given in Appendix A, as

$$f_{N,\nu'}(v)=\sum_{l=0}^{3(N-1)/2} a_{l,\nu'}\Theta_o\begin{bmatrix}(1+\nu'+2l)/3N\\ 0\end{bmatrix}(3Nv,3N\tau), \tag{5.22}$$

where

$$\begin{aligned} \Theta_o\begin{bmatrix}(1+\nu'+2l)/3N\\ 0\end{bmatrix}(3Nv,3N\tau) = &\ \Theta\begin{bmatrix}(1+\nu'+2l)/3N\\ 0\end{bmatrix}(3Nv,3N\tau) \\ &-\Theta\begin{bmatrix}(1+\nu'+2l)/3N\\ 0\end{bmatrix}(-3Nv,3N\tau). \end{aligned} \tag{5.23}$$

We now use the form (5.22) in the difference Eq. (5.11) and then use (A11) to obtain

$$\begin{aligned} &\sum_{l=0}^{3(N-1)/2} a_{l,\nu'}\{1+e^{i2\pi(1+\nu'+2l)/3}+e^{-i2\pi(1+\nu'+2l)/3}\}\Theta_o \\ &\qquad\times\begin{bmatrix}(1+\nu'+2l)/3N\\ 0\end{bmatrix}(3Nv,3N\tau)=0. \end{aligned} \tag{5.24}$$

This equation is satisfied if (for $\nu'=0,1$)

$$a_{l,\nu'}=0 \quad \text{for} \quad 1+\nu'+2l\equiv 0 \ \text{mod}3. \tag{5.25}$$

Then in (5.22) for $l \equiv 0 \bmod 3$ we use (A15) to send the characteristic $(1+\nu'-2l)/3N$ to $-(1+\nu'-2l)/3N$, and $a_{l,\nu'} \to -a_{l,\nu'}$. Then if we use (A14) and (A15) and send $a_{l,\nu'} \to a_{l-(N-1)/2,\nu'}$ we may reorder the terms in (5.22) to write the expression for $f_N(v)$ in the more elegant form

$$f_{N,\nu'}(v) = \sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,\nu'} \Theta_o \begin{bmatrix} (6l-1-\nu')/3N \\ 0 \end{bmatrix} (3Nv, 3N\tau). \tag{5.26}$$

To determine the remaining N coefficients $\tilde{a}_{l,\nu'}$ we note that from the definition (5.10) that $f_{N,\nu'}(v)$ must have an Nth order zero at $v=0$ and $v=iK'$ and thus the first $N-1$ derivatives of $f_{N,\nu'}(v)$ must vanish at $v=0,\ iK'$.

To satisfy the condition at $v=0$ we note that because $f_{N,\nu'}(v)$ is odd this requires that the odd derivatives up to order $N-2$ must vanish and therefore we have the $(N-1)/2$ equations

$$\sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,\nu'} \Theta_o^{(2m-1)} \begin{bmatrix} (6l-1-\nu')/3N \\ 0 \end{bmatrix} (0, 3N\tau) = 0 \tag{5.27}$$

for $m=1,2,\ldots,(N-1)/2$ where the superscript $2m-1$ indicates the $2m-1$ derivative with respect to v.

To satisfy the condition at $v=iK'$ we note that for any ϵ

$$\begin{aligned} &\Theta_o \begin{bmatrix} \epsilon \\ 0 \end{bmatrix} (3N(v+iK'), 3N\tau) \\ &\quad = q^{-3N/4} e^{-\pi i 3Nv/2K} \Theta_o \begin{bmatrix} 1+\epsilon \\ 0 \end{bmatrix} (3Nv, 3N\tau). \end{aligned} \tag{5.28}$$

Therefore $e^{\pi i 3Nv/2K} f_{N,\nu'}(v+iK')$ is an odd function of v, which is given by

$$\begin{aligned} &e^{\pi i 3Nv/2K} f_{N,\nu'}(v+iK') \\ &\quad = q^{-3N/4} \sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,\nu'} \Theta_o \begin{bmatrix} 1+(6l-1-\nu')/3N \\ 0 \end{bmatrix} (3Nv, 3N\tau). \end{aligned} \tag{5.29}$$

This odd function has a zero of order N at $v=0$ and thus we find the companion equations to (5.27) of

$$\sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,\nu'} \Theta_o^{(2m-1)} \begin{bmatrix} 1+(6l-1-\nu')/3N \\ 0 \end{bmatrix} (0, 3N\tau) = 0 \qquad (5.30)$$

for $m = 1, 2, \ldots, (N-1)/2$

The N coefficients $\tilde{a}_l$ of the expansion (5.26) are thus determined from $N-1$ homogeneous equations (5.27) and (5.30) and hence the existence of solutions for the TQ equation for the eigenvalue (5.1) of $\mathbf{T}(v)$ has been demonstrated.

Finally we show that

$$q^{3N/4} e^{\pi i 3Nv/2K} f_{N,0}(v+iK') = f_{N,1}(v), \qquad (5.31)$$

from which if follows that

$$q^{3N/4} e^{\pi i 3Nv/2K} Q_{N,\nu'=0}(v+iK') = Q_{N,\nu'=1}(v). \qquad (5.32)$$

The result (5.31) follows from (5.29) if we can show that

$$\sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,0} \Theta_o \begin{bmatrix} 1+(6l-1)/3N \\ 0 \end{bmatrix} (3Nv, 3N\tau) = \sum_{l=-(N-1)/2}^{(N-1)/2} \tilde{a}_{l,1} \Theta_o \begin{bmatrix} (6l-2)/3N \\ 0 \end{bmatrix} (3Nv, 3N\tau). \qquad (5.33)$$

If $(N-1)/2$ is even this will follow if we can show that

$$a_{l,0} = a_{(N-1)/2-l+1,1} \quad \text{for} \quad 1 \leqslant l \leqslant (N-1)/2, \qquad (5.34)$$

$$a_{l,0} = a_{-l-(N-1)/2,1} \quad \text{for} \quad -(N-1)/2 \leqslant l \leqslant 0 \qquad (5.35)$$

and (5.34) and(5.35) are easily seen to follow from (5.27) to (5.30). The case $(N-1)/2$ odd is treated in a similar manner and thus (5.31) is established.

The linear equations (5.27) and (5.30) can of course be solved as determinants. It would be much more helpful, however, if these determinants could be evaluated in some simpler form. Such a simplification, if it exists, unfortunately involves identities in theta constants which do not seem to be in the literature. We thus content ourselves here with the remark that for $N=3$ we have shown that the coefficients $\tilde{a}_l$ are modular forms for the subgroup of the modular group defined by the transformation

$$\tau \to \frac{a\tau+b}{c\tau+d}, \tag{5.36}$$

where

$$a \equiv 1 \ \mathrm{mod} 6N, \quad b \equiv 0 \ \mathrm{mod} 2, \quad c \equiv \ 0 \ \mathrm{mod} 3N, \quad d \equiv 1 \ \mathrm{mod} 6N \tag{5.37}$$

6. THE SIX VERTEX LIMIT

We may now consider limit $q \to 0$ where the eight vertex model reduces to the six vertex model. In this limit

$$\eta \to \frac{m\pi}{2L} \tag{6.1}$$

and in the XXZ Hamiltonian (1.24) we have

$$\Delta = \cos \frac{m\pi}{L}. \tag{6.2}$$

6.1. N Odd, $\eta = mK/L$ with m Odd and Generic η

In the eight vertex model we saw in Section 2 for N odd, $\eta = mK/L$ with m odd and in Section 3 for generic η that all states occur in pairs such that for every set of roots v_k there is a companion set of roots $v_k + iK'$ and that both eigenvalues of $Q_{72}(v)$ correspond to the same (doubly degenerate) eigenvalues of $T(v)$. This double degeneracy of all transfer matrix eigenvalues for chains of odd length follows from the spin inversion symmetry of the Boltzmann weights.

Consider the fundamental region to be given by

$$0 \leqslant \mathrm{Re}(v) < 2K \quad \text{and} \quad -K'/2 \leqslant \mathrm{Im}(v) < 3K'/2. \tag{6.3}$$

Then the pairing of solutions v_k means that for every solution with n roots in the region

$$-K'/2 \leqslant \mathrm{Im}(v) < K'/2 \tag{6.4}$$

and $N-n$ roots in the region

$$K'/2 \leqslant \mathrm{Im}(v) < 3K'/2, \tag{6.5}$$

there will be a corresponding solution with $N-n$ roots in (6.4) and n roots in (6.5). Numerical studies indicate that as $q \to 0$ that the n roots in (6.4) stay a finite distance from $v=0$ and the $N-n$ roots in (6.5) stay a finite distance from $v=iK'$. Therefore in the limit $q \to 0$ with v held fixed the quasi-periodic functions which are the eigenvalues of $Q_{72}(v)$ and which always have N zeroes in the fundamental region reduce to trigonometric polynomials which have n zeroes in the strip $0 \leqslant \mathrm{Re}(v) < \pi$ where n can take ALL values from 0 to N. These n zeroes satisfy the Bethe's equation obtained from taking the $q \to 0$ limit of (2.14)

$$\left(\frac{\sin(v_l - m\pi/2L)}{\sin(v_l + m\pi/2L)}\right)^N = e^{2\pi i v m/L} \prod_{\substack{j=1\\ j\neq l}}^{n} \frac{\sin(v_l - v_j - m\pi/L)}{\sin(v_l - v_j + m\pi/L)}. \tag{6.6}$$

We note in particular the following points:

(1) To every solution of (3.3) with $n \leqslant (N-1)/2$ roots there exists a "companion" solution with $N-n$ roots which gives the same eigenvalue of the transfer matrix $T(v)$. This is the phenomenon discovered by Pronko and Stroganov[21] in the case where γ in the XXZ Hamiltonian (1.24) is irrational. Examples of such pairs are given in Table VI.

(2) There are never any solutions with infinite roots or L strings such as occur for even N as seen in refs. 8 and 11

(3) The limit of the Bethe equation (2.10) is formally the same as the limit of (2.14) but eq. (2.10) does not allow solutions with $n \geqslant (N+1)/2$.

This resolution of the question of the existence of solutions to the Bethe's equation (6.6) with $n > N/2$ with odd N is very different from the resolution for even N presented in ref. 8 which involves both L strings and roots at infinity.

6.2. $\eta = mK/L$ with m Even

For the eight vertex model with N odd and m even we saw in Section 4 that there are two types of solutions. Those with L strings with $n_L \geqslant 1$

and with $2n_B$ roots, which occur in pairs $v_k^B,\ v_k^B+iK'$ and those with neither L strings nor paired roots, where there are two solution for $Q_{N,\nu'}(v)$ where the solutions for $\nu'=0,1$ are paired under $v\rightarrow v+iK'$.

The first type of solution is of the form previously seen for N even and the six vertex limit of these solutions will in general contain L strings which would seem to be in contradiction with the statement of Section 4 of ref. 8 that "for N odd no problems appeared" in taking the limit $H\rightarrow 0$.

The second sort of solution will behave in the six vertex limit in a fashion similar to the case of m odd which is illustrated by considering the limit $q\rightarrow 0$ in the solution for $f_{N,\nu'}(v)$. When $q\rightarrow 0$ we use (A17) to find from (5.26) that

$$\lim_{q\rightarrow 0} f_{N,\nu'}(v)=\sum_{l=-(N-1)/2}^{(N-1)/2} b_{l,\nu'}2i\sin(6l-1-\nu')v, \tag{6.7}$$

where

$$b_{l,\nu'}=\lim_{q\rightarrow 0}\tilde{a}_{l,\nu'}q^{(6l-1-\nu')^2/36N^2} \tag{6.8}$$

and from (5.27) the $b_{l,\nu'}$ satisfy

$$\sum_{l=-(N-1)/2}^{(N-1)/2} b_{l,\nu'}(6l-1-\nu')^{2m-1}=0. \tag{6.9}$$

For the remaining Eq. (5.30) we use again (A17) to find for small q that

$$\begin{aligned}&\sum_{l=-(N-1)/2}^{0} b_{l,\nu'}q^{[3N+2(6l-1-\nu')]/12N}(3N+6l-1-\nu')^{2m-1}\\&+\sum_{l=1}^{(N-1)/2} b_{l,\nu'}q^{[3N-2(6l-1-\nu')]/12N}(-3N+6l-1-\nu')^{2m-1}=0.\end{aligned} \tag{6.10}$$

We note that in (6.10) there are two types of terms: those where the coefficient of $b_{l,\nu'}$ vanishes as $q\rightarrow 0$ which impose no constraint on $b_{l,\nu'}$ and those where the coefficient of $b_{l,\nu'}$ diverges as $q\rightarrow 0$ which forces the $b_{l,\nu'}$ to vanish. We thus find for $\nu'=0$

$$\text{for } \frac{N-1}{4} \text{ integer } \quad b_{l,0}\neq 0 \quad \text{only for } -\frac{N-1}{4}\leqslant l\leqslant \frac{N-1}{4} \tag{6.11}$$

and

$$\text{for } \frac{N-3}{4} \text{ integer } \quad b_{l,0}\neq 0 \quad \text{only for } -\frac{N-3}{4}\leqslant l\leqslant \frac{N-3}{4}, \tag{6.12}$$

while
for $\nu'=1$

$$\text{for } \frac{N-1}{4} \text{ integer } \quad b_{l,0}\neq 0 \quad \text{only for } -\frac{N-1}{4}+1\leqslant l\leqslant \frac{N-1}{4} \tag{6.13}$$

and

$$\text{for } \frac{N-3}{4} \text{ integer } \quad b_{l,0}\neq 0 \quad \text{only for } -\frac{N-3}{4}\leqslant l\leqslant \frac{N-3}{4}+1. \tag{6.14}$$

For $\nu'=0$ and $(N-1)/4$ an integer we use (6.11) in (6.7) and (6.9) with $l=(N-1)/4-k$ and $b_{l,0}=\alpha_k$ to find

$$\lim_{q\to 0} f_{N,0}(v)=-2i\sum_{k=0}^{(N-1)/2}\alpha_k \sin(6k-3(N-1)/2+1) \tag{6.15}$$

with

$$\sum_{k=0}^{(N-1)/2}\alpha_k(6k-3(N-1)/2+1)^{2m-1}=0 \tag{6.16}$$

and for $\nu'=1$ and $(N-3)/4$ an integer we use (6.14) in (6.7) and (6.9) with $l=k-(N-3)/4$ and $b_{l,0}=\alpha_k$ to find

$$\lim_{q\to 0} f_{N,1}(v)=2i\sum_{k=0}^{(N-1)/2}\alpha_k \sin(6k-3(N-1)/2+1), \tag{6.17}$$

where again α_k satisfies (6.16).

For the six vertex model the problem of finding a solution of the TQ equation with $(N-1)/2$ Bethe roots for the six vertex eigenvalue $T_6(v) = (\sin v)^N$ has been previously studied by Stroganov.[23] If we note that the function $f_{\rm strog}(v)$ of ref. 23 satisfies

$$f_{\rm strog}(v+\pi) = (-1)^{(N+1)/2} f_{\rm strog}(v), \tag{6.18}$$

instead of (5.12) it is seen that (6.15)–(6.17) agrees with $f_{\rm strog}(v)$ for all odd N.

For $\nu' = 0$ and $(N-3)/4$ an integer we use (6.12) in (6.7) and (6.9) with $l = k - (N-3)/4$ and $b_{l,0} = \beta_k$ to find

$$\lim_{q\to 0} f_{N,0}(v) = 2i \sum_{k=0}^{(N-3)/2} \beta_k \sin(6k - 3(N-1)/2 + 2), \tag{6.19}$$

where

$$\sum_{k=0}^{(N-3)/2} \beta_k (6k - 3(N-1)/2 + 2)^{2m-1} = 0 \tag{6.20}$$

and for $\nu' = 1$ and $(N-1)/4$ an integer we use (6.13) with $l = (N-1)/4 - k$ and $\beta_k = b_{l,1}$ to find

$$\lim_{q\to 0} f_{N,1}(v) = 2i \sum_{k=0}^{(N-3)/2} \beta_k \sin(6k - 3(N-1)/2 + 2), \tag{6.21}$$

where the β_k still satisfy (6.20). These functions are of the form of the "second solution" g(v) eqn. (12) of Stroganov[23] except the upper limit of (6.19) and (6.21) is $(N-3)/2$ instead of $(N-1)/2$.

7. DISCUSSION

The difficulty of discussing what in the literature is called "Baxter's Q" is that over the years three different objects have been defined which have all been denoted by the same symbol and all of which satisfy a "TQ" equation (1.7).

7.1. Three Definitions of *Q*

The first definition of Q is the matrix, which we have here called $\mathbf{Q}_{72}(v)$, defined in[2] only when η satisfies the root of unity condition (1.4). The definition requires an auxiliary matrix $\mathbf{Q}_{72,R}(v)$ to be nonsingular and in paper I[15] we found that if m is even and N is odd or when m and N are both even and $N \geqslant L-1$ then the non-singularity assumption on $\mathbf{Q}_{72,R}(v)$ fails. The matrix $\mathbf{Q}_{72}(v)$ commutes with $\mathbf{S}$ but not with $\mathbf{R}$ and satisfies the quasi-periodicity relations (1.13) and (1.14). We have seen repeatedly in ref. 15 and in this present paper that the eigenvalues of $\mathbf{Q}_{72}(v)$ are in general of the form (2.1)

$$Q_{72}(v) = \mathcal{K}(q, v_k)\exp(-i\nu\pi v/2K)\prod_{j=1}^{N} H(v - v_j), \tag{7.1}$$

where the v_j satisfy the sum rules (2.4) and (2.5). The matrix $\mathbf{Q}_{72}(v)$ has no degenerate eigenvalues even though many of the eigenvalues of $\mathbf{T}(v)$ are degenerate.

The second definition of Q is the matrix we have here called $\mathbf{Q}_{73}(v)$ defined in Section 6 of ref. 3 and in Section 10.5 of ref. 6 for generic values of η not just those satisfying the root of unity condition (1.4). This definition applies only for N even. The matrix $\mathbf{Q}_{73}(v)$ commutes with both $\mathbf{S}$ and $\mathbf{R}$ and it is shown in (10.5.43) of ref. 6 that it satisfies the quasi-periodicity relations (1.16 and (1.17). All eigenvalues of $\mathbf{Q}_{73}(v)$ are of the form (123) of ref. 8 [and (10.6.8) of ref. 6]

$$Q_{73}(v) = e^{2i\tau v}\prod_{j=1}^{N/2} h(v - v_j), \tag{7.2}$$

where the v_j satisfy the sum rule (125) of ref. 8 [and (10.6.7) of ref. 6] which depends on the eigenvalues of $\mathbf{S}$ and $\mathbf{R}$. The matrices $\mathbf{Q}_{72}(v)$ and $\mathbf{Q}_{73}(v)$ are not similar to each other and the eigenvalues of the form (7.2) with generic η cannot in general specialize to (7.1) when η is a root of unity.

There is yet one more definition of Q given in (1.24) of ref. 5 and (132) of ref. 8. In this definition $Q(v)$ is (a set of) scalar function(s) defined by

$$Q(v) = \prod_{j=1}^{n_B} h(v - v_j), \tag{7.3}$$

where [see (1.18)] $N=2n_B+L_{73}n_L$, the v_j are the solutions of the Bethe's equation (2.10) which is derived for the related solid on solid model defined for η satisfying the root of unity condition (1.4) and there is no sum rule on the v_j unless $n_L=0$ (see footnote 15 of ref. 8). The properties of this scalar $Q(v)$ under translations $v\rightarrow v+2K$ and $v\rightarrow v+iK'$ (given in (134) of ref. 8) are obtained directly from (A8) and in contrast with the quasi-periodicity properties (1.16) and (1.17) will explicitly involve the sum of the roots $\sum_k v_k$ which cannot be eliminated due to the lack of a sum rule.

From the definition (7.3) of $Q(v)$ a matrix is constructed in (1.29) of ref. 5 with the assumption that all eigenvalues of $\mathbf{T}(v)$ can be obtained from these scalar functions $Q(v)$ which are now regarded as eigenvalues of some matrix. However all eigenvalues of $\mathbf{T}(v)$ for odd N and many eigenvalues for even N are at least doubly degenerate. All degenerate eigenvalues of $\mathbf{T}(v)$ have the same n_B Bethe roots v_j in (7.3) and thus this construction must lead to a matrix $\mathbf{Q}(v)$ which also has degenerate eigenvalues and hence the eigenvalues of this $\mathbf{Q}(v)$ contain no information about the multiplicities of the eigenvalues of $\mathbf{T}(v)$.

7.2. Quasiperiodicity Conditions

All three definitions of $Q(v)$ discussed above solve the same TQ equation. They differ only in which quasi-periodicity conditions (3.1)–(3.2) or (1.16)–(1.17) are imposed. These may be thought of as boundary conditions for the TQ difference equation. We emphasize that at roots of unity the TQ equation is in general not sufficient to determine all the eigenvalues of $\mathbf{Q}(v)$ but is it a necessary condition that any $\mathbf{Q}(v)$ matrix must satisfy.

The quasi-periodicity conditions (3.1) and (3.2) comes from the assumption that $\mathbf{Q}(v)$ commutes with $\mathbf{S}$ and that all eigenvalues of $\mathbf{Q}(v)$ are quasiperiodic entire functions with N zeroes in the fundamental region of the Boltzmann weights (1.2). If $\mathbf{Q}(v)$ has no degenerate eigenvalues and if its rank is 2^N then the eigenvalues of $\mathbf{Q}(v)$ will characterize all the (possibly degenerate) eigenvalues of $\mathbf{T}(v)$.

The quasi-periodicity conditions (1.16) and (1.17) imposes the additional restriction that the zeroes must occur in pairs v_k and v_k+iK' (which is possible only for N even). These quasi-periodicity conditions have been extensively discussed in the literature. They are the conditions used in refs. 3–5, 6 and 15 to solve the eight vertex model with N even for η either generic or a root of unity. A mathematical study of the solutions of the TQ equation (1.7) with these quasi-periodicity conditions was made in ref. 20.

The principle "new development" of this and the previous paper[15] is the introduction of the quasi-periodicity conditions (3.1) and (3.2) which are less restrictive than the quasi-periodicity conditions (1.16) and (1.17) satisfied by $\mathbf{Q}_{73}(v)$.

The quasi-periodicity condition (3.2) was first seen in ref. 15 to apply to $\mathbf{Q}_{72}(v)$ defined for roots of unity. For odd N (3.2) must be used for all η because for odd N $\mathbf{R}$ and $\mathbf{S}$ do not commute and it is impossible to diagonalize $\mathbf{Q}(v)$, $\mathbf{S}$ and $\mathbf{R}$ at the same time. Hence it follows that (1.17) cannot be applied in this case.

7.3. Second Solutions

The TQ equation is a second order difference equation and as such it is expected to have two independent solutions. For the quasi-periodicity conditions (3.1) and (3.2) this is quite obvious because if $\mathbf{Q}(v)$ satisfies the TQ equation (1.7) then $e^{i\pi Nv/2K}\mathbf{Q}(v+iK')$ satisfies the same equation. Therefore as long as $\mathbf{Q}(v)$ and $e^{i\pi Nv/2K}\mathbf{Q}(v+iK')$ are linearly independent the TQ (1.7) equation will have two solutions. For odd N these two solutions are characterized by the two values of quantum number $\nu' = 0, 1$ whereas for even N both solutions have the same value of ν'. These two linearly independent solutions for $\mathbf{Q}(v)$ correspond to the degeneracy of the eigenvalues of $\mathbf{T}(v)$ under spin reversal.

The only situation where this linear independence fails is when the quasi-periodicity (1.17) holds. This can only happen for N even and the mathematical existence of a second solution for η not a root of unity has been discussed in ref. 20. This solution is the analogue of a second solution of linear differential equations with equal roots of the indicial equation. These solutions will not be entire functions of the variable v and hence will not be allowed solutions for the eight vertex model. This agrees with the fact that for N even and η not a root of unity the eigenvalues of $\mathbf{T}(v)$ are not degenerate.[3–5]

7.4. The Six Vertex Limit

For odd N the six vertex limit of the TQ equation with the quasiperiodicity conditions (3.2) is easily taken. For m odd the two independent solutions $\mathbf{Q}(v)$ and $e^{i\pi Nv/2K}\mathbf{Q}(v+iK')$ smoothly go to the solutions given by Baxter,[8] Pronko and Stroganov[21] and Korff[18] where one solution has less that $N/2$ roots and one solution has more that $N/2$ roots. The corresponding eigenvalues of the transfer matrix are doubly degenerate under spin inversion. The six vertex case for m even is treated in ref. 18.

For even N the limit is more subtle. When η is not a root of unity and N is even the quasiperiodicity relation (1.17) holds. This quasiperiodicity relation does not allow for spin doublets. The eigenvectors of $\mathbf{Q}(v)$ are eigenvectors of $\mathbf{R}$ and the eigenvalues which correspond to the two different eigenvalues of $\mathbf{R}$ are different.[3–6] In the six vertex limit the eigenvectors of the eight vertex model will go to linear combinations of Bethe vectors which are eigenvectors of S^z. For generic η the six vertex model has two solutions[9] $\mathbf{Q}(v)$ of the TQ equation but one of these violates the analyticity assumptions needed for $\mathbf{Q}(v)$ to be relevant for the six vertex model. A resolution of this in terms of roots at infinity is given in ref. 8.

The case of even N and η a root of unity is also discussed in refs. 8 and 18.

7.5. Conclusion

In this paper we have seen for odd N with $\eta = mK/L$ and m odd that the zeroes of $\mathbf{Q}_{72}(v)$ do not satisfy what is usually called "Bethe's" equation (2.10) but instead satisfy the more general generic equation for roots (3.3). We conjecture that for irrational η/K a matrix $\mathbf{Q}(v)$ exists which satisfies the TQ equation (1.7) with (1.5), (1.6) and the quasi-periodicity conditions (3.1) and (3.2). The roots of this conjectured matrix will also satisfy (3.3). For neither $\eta = mK/L$ with m odd nor for irrational η/K will there be any L strings or paired roots $v_k,\ v_k + iK.'$

For N odd and $\eta = 2mK/L$ no $\mathbf{Q}(v)$ is known to exist but the limit $\eta \to 2mK/L$ was studied above and it was seen that there are two distinct classes of eigenvalues $Q(v)$: (1) those which do not have L strings or paired roots and (2) those which have an odd number of L strings $n_L \geqslant 1$ and the remaining roots all occur in pairs. This second class of eigenvalues are obtained from the eigenvectors of $T(v)$ computed in refs. 3–5 from considering the related SOS model. We conjecture that a $\mathbf{Q}(v)$ matrix exists which will incorporate both the states computed by Baxter.[3–5] and those which are not computed by those methods. The existence of L strings and degenerate eigenvalues of $\mathbf{T}(v)$ implies that this $\mathbf{Q}(v)$ will not be unique.

APPENDIX A: PROPERTIES OF THETA FUNCTIONS

The definition of Jacobi theta functions of nome q is

$$H(v) = 2\sum_{n=1}^{\infty}(-1)^{n-1}q^{(n-\frac{1}{2})^2}\sin[(2n-1)\pi v/2K], \tag{A1}$$

$$\Theta(v)=1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(nv\pi/K), \tag{A2}$$

where K and K' are the standard elliptic integrals of the first kind and

$$q=e^{-\pi K'/K}=e^{i\pi\tau}. \tag{A3}$$

These theta functions satisfy the quasi-periodicity relations

$$H(v+2K) = -H(v), \tag{A4}$$
$$H(v+2iK') = -q^{-1}e^{-\pi iv/K}H(v), \tag{A5}$$
$$\Theta(v+2K) = \Theta(v), \tag{A6}$$
$$\Theta(v+2iK') = -q^{-1}e^{-\pi iv/K}\Theta(v). \tag{A7}$$

From (A2) we see that $\Theta(v)$ and $H(v)$ are not independent but satisfy

$$\begin{aligned}\Theta(v\pm iK') &=\pm iq^{-1/4}e^{\mp\frac{\pi iv}{2K}}H(v),\\ H(v\pm iK') &=\pm iq^{-1/4}e^{\mp\frac{\pi iv}{2K}}\Theta(v).\end{aligned} \tag{A8}$$

Theta functions with characteristics ϵ and ϵ' are defined as

$$\begin{aligned}\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v,\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi(n+\epsilon/2)^2\tau}e^{\pi i(n+\epsilon/2)(u/K+\epsilon')}\\ &= \sum_{n=-\infty}^{\infty} q^{(n+\epsilon/2)^2}e^{\pi i(n+\epsilon/2)(u/K+\epsilon')}.\end{aligned} \tag{A9}$$

In this notation

$$H(v)=-\Theta\begin{bmatrix}1\\1\end{bmatrix}(v),\quad \Theta(v)=\Theta\begin{bmatrix}0\\1\end{bmatrix}. \tag{A10}$$

These theta functions satisfy the quasi-periodicity conditions

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v+2K,\tau)=e^{i\pi\epsilon}\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v,\tau) \tag{A11}$$

and

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v+2iK',\tau)=e^{\pi i\epsilon'}q^{-1}e^{-i\pi u/K}\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v,\tau). \tag{A12}$$

These quasi-periodicity properties guarantee that $\Theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](v,\tau)$ has exactly one zero in the fundamental region

$$0\leqslant \mathrm{Re}(v)<2K, \qquad 0\leqslant \mathrm{Im}(v)<2K'. \tag{A13}$$

The functions have the further properties that

$$\Theta\begin{bmatrix}\epsilon\pm 2m\\ \epsilon'\pm 2n\end{bmatrix}(v,\tau)=e^{-i\pi n\epsilon}\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v,\tau), \tag{A14}$$

$$\Theta\begin{bmatrix}-\epsilon\\ -\epsilon'\end{bmatrix}(v,\tau)=\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-v,\tau), \tag{A15}$$

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v+iK')=q^{-1/4}e^{-i\pi\epsilon'/2}e^{-i\pi v/2K}\Theta\begin{bmatrix}\epsilon+1\\ \epsilon'\end{bmatrix}(v,\tau) \tag{A16}$$

and as $q\rightarrow 0$ with $|\epsilon|<1$ we have

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(v,\tau)\rightarrow q^{\epsilon^2/4}e^{\pi i\frac{\epsilon}{2}(v/K+\epsilon')}. \tag{A17}$$

Further discussion and properties can be found in reference books on theta functions such as refs. 16 and 19.

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